Education

Eric Wang

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1 Purpose of Education

The ultimate purpose of education is to promote curiosity and a desire to gain knowledge. This very general goal clearly cannot be approached directly - how can schools somehow force their students to become interested in something they are not? Of course, it can poke them with an incentive (grading system, for example) to study, but that alone doesn't achieve the purpose. I claim that by equipping students with the ability to solve problems, the ultimate purpose will also be achieved. That is, students who are good problem solvers are also naturally inclined to be motivated to gain knowledge on their own. Roughly, this is because the methods by which the education system can make students better problem solvers (discussed later in this document) should also inspire curiosity. Further, this more specific goal of problem solving is more approachable than the general one. But before this can be accomplished, certain changes in attitude from both teachers' and students' perspectives must occur.

2 Common Complaints

I have often heard students complain about the usefulness of the classes they are forced to take: something along the lines of "Why must I take class X if it has nothing to do with my future job Y?" or "How am I ever going to apply this in my life?" Such complaints reflect a narrow view of the purpose of education. For example, many students who study math in school do not understand its importance. After all, how can topics such as similar triangles or quadratic equations possibly be applied to one's life? But this grievance overlooks the fact that the study of mathematical subjects such as geometry are just means to an end. They are not necessarily important on their own; indeed, Euclidean geometry, for example, is rather esoteric in the context of modern math. Instead, studying elementary math offers an approachable entry into important elements not specific to math - for example, proof writing (a proof is nothing but a rigorous explanation of how a result is derived with some mathematical language thrown in). The ability to produce a proof (that is, to formulate one's thoughts into precise language) demonstrates clearness and organization of thinking. People proficient in proof writing are more likely to deliver coherent presentations, give lucid explanations, and avoid logical fallacies. Such qualities are clearly indispensable in life. In general, someone who complains about a subject's limited applicability in their life probably overlooks the underlying skills gained from studying it. This refers back to the purpose of education as proposed earlier: it uses specific subjects to guide students to a larger, more general objective.

The above arguments only hold in a situation where the curriculum enters a subject deeply enough to actually make an impact on students' problem solving abilities. In the example of proof writing above, a typical geometry course (the first math course where students encounter the idea of a proof) expects of students no more than a three line "proof" of some ridiculously trivial fact. Of course, there is always the dilemma between implementing a curriculum that emphasizes breadth of topics versus depth of a particular subject. The map of subjects and subtopics can be likened to a directory; for example

$$\text{Math} \to \text{Algebra} \to \text{Inequalities} \to \text{Smoothing} \to \text{Majorization} \to \cdots$$

Standard curricula usually only develop a surface level overview of the subject; they extend only a few levels into a path in the directory. A shallow understanding of a subject will not improve someone's ability to solve problems. As a result, it is left to the students to advance their skills outside of school. But then the question is, what purpose does education serve in this case? On the other hand, it is not necessarily desirable to change the curriculum to make it more comprehensive and in-depth for each subject. But holding the curriculum constant, there are still ways teachers can change their teaching style to more effectively instill the problem solving ability in their students.

3 Intuition and Algorithmic Thinking

The one flaw of teaching that will most likely turn students away from rich and interesting subjects is the lack of intuition. Without an intuitive understanding of the subject, students can only apply what they have learned to problems they have seen before; they will not be able to generalize because they were taught to execute a specific algorithm to solve a specific problem. Additionally, sharing intuition prompts discussion and sparks curiosity. Students will try to find different ways to approach problems instead of running the procedure their teacher drilled into them. And if that method fails, they will not be motivated to try again with a different one. This robotic, algorithmic thinking becomes an obstacle in the way of effective problem solving. Here's an example with solving quadratics.

You are given a quadratic equation in x that you wish to solve:

$$ax^2 + bx + c = 0$$

1. Divide by the leading coefficient

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2. Subtract the constant term

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Add $\frac{b^2}{4a^2}$ on both sides

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Factor the left hand side as a square

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

5. Take the square root on both sides, remembering the plus/minus

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

6. Subtract the remaining term to isolate x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In many ways, the above algorithm is not satisfactory to a student. Specific questions might include, how did that square magically appear in step 4, what if that number under the square root is negative, or what if one of the coefficients is zero? But overall, how would someone have come up with this way to solve quadratics? Is there a more general technique that can be used for other equations, perhaps higher degree polynomials?

Many such questions remain unanswered. But even worse, they probably also go unasked. In order that students feel the need to ask questions, they must feel that something is incomplete about the way the topic was explained to them. However, if they have always been taught in this mechanical way, they will not realize that something is in fact wrong. Teaching algorithms without intuition is hugely detrimental to both students' problem solving abilities and promoting interest in the subject.

Giving intuition means not only to present the solution to a problem, but also to explain how one could have conceived it. It means to translate the rigor of a textbook into language more easily understood. It means to explain how to link different ideas together - how to use problems you know how to solve to solve unknown ones. It means not only to present a theorem or technique but also to show where it should be used and how someone could have thought to use it in that situation. It means to communicate how to think about something. Here's a concrete example:

Consider the Cauchy-Schwarz inequality, which states that for sequences $\{a_i\}$ and $\{b_i\}$ of nonnegative real numbers, the following inequality holds:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

Now prove the so-called Titu's Lemma, which states that for sequences of nonnegative reals $\{a_i\}$ and $\{b_i\}$, we have

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

After manipulating the Cauchy inequality for a while, you might stumble across the substitution

$$(a_i, b_i) \to \left(\frac{a_i}{\sqrt{b_i}}, \sqrt{b_i}\right)$$

which does indeed solve the problem. However, that substitution wasn't exactly obvious. But suppose you were given the intuition that Cauchy should be interpreted in this form:

$$(a_1 + a_2 + \cdots + a_n)(b_2 + b_2 + \cdots + b_n) \geq (\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n})^2$$

In words, the product of the sum of two sequences is bounded from below by the square of the sum of the geometric means of corresponding terms. Clearly, it is equivalent to the original form. But with this interpretation, Titu's Lemma suddenly becomes obvious. (If you don't see it, try multiplying $b_1 + b_2 + \cdots + b_n$ on both sides.)

This demonstrates the importance of having intuition of how to apply certain knowledge. But some intuition only comes from having experience solving many problems; such advice cannot simply be communicated because it is more similar to a feeling than a technique. To motivate such intuition, one must distinguish between problems and exercises and select an appropriate combination of them.

4 Problems and Exercises

Suppose you have just learned about quadratics (the formula was derived in the previous section). In order to familiarize yourself, you will need a few examples like the following.

Example 1. *Solve for x:* $2x^2 = 8x - 7$

But if that's the extent of the practice done with quadratics, it will be difficult to solve harder tasks that differ from the above example even slightly. In order to advance, you would need to consider more involved tasks such as this.

Example 2. *Determine the range of a for which $x^2 + (a-1)x + 1 = 0$ has real solutions.*

Of course, difficulty is just a function of the skill of the problem-solver. Once you are confident and can easily solve the above examples, you may want to approach exceptionally difficult problems.

Example 3. *Let f and g be quadratics with real coefficients. Suppose that if the number $g(k)$ is an integer for a positive k, then so is $f(k)$. Prove that there exist integers m and n such that $f(x) = mg(x) + n$ for all real x.*[1]

The point here is that there is a difference between problems and exercises. To someone who has just learned about quadratics, the first example is definitely an exercise, while the second is perhaps more of a problem. An exercise seeks to familiarize the student with a new concept. It often features a direct application of the material learned. In contrast, problems are intended to help the student discover less obvious ways to apply what they learned - in other words, to promote creative thinking. If the student knows exactly how to proceed, then the task cannot be considered a problem - it is merely an exercise. Exercises are indeed necessary; they are effective for easing students into new concepts. But problems are the ones that ultimately impact the student's problem solving skills.

Unfortunately, in most cases the math curriculum focuses mostly, if not entirely, on exercises. The exercises section in textbooks sometimes even has headings that

[1] Bulgaria 1996

instruct the student how to proceed: solve by factoring, by completing the square, by quadratic formula, etc. This reflects the emphasis on algorithmic thinking; the student has seen the exact exercise before with a change of a few numbers and mechanically regurgitates the algorithm. This brings about the opposite effect to what education should have. In order to enhance students' problem solving abilities, they must approach both problems and exercises.

5 Solutions

I am optimistic about the overall capacity for education to positively impact students. Of course, there are shortcomings as outlined previously, but those exist in any institution. The most dangerous outcome for education is if it starts to focus on the wrong things and on what is not important. Here's a recent example (note that I am not criticizing the news article, just the pointless topic of debate).[2] Armed with a good education, students will be able to solve the problems outlined above. Perhaps they will disagree on the problem itself and decide that the problem isn't as naive as that I proposed earlier. In view of this Seattle Times article and the possibility of changing education for the worse, maybe the problem isn't entirely contained in the realm of education itself. It seems that identifying the relevant issues is a problem itself. Regardless of external changes, we should remember that the ultimate purpose of education remains invariant.

[2] Takahama, Elise, "Is math racist? New course outlines prompt conversations about identity, race in Seattle classrooms"